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The development of Long's vortex

By P. G. DRAZIN, W. H. H. BANKS AND M. B. ZATURSKA
School of Mathematics, University of Bristol, Bristol BS8 1TW, UK

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This paper describes the solution of Long's problem for steady rotationally symmetric swirling jets in a uniform viscous fluid. Long found these vortices in 1958 by assuming a similarity form of solution, and in 1961 solved the consequent problem in the boundary-layer limit, finding dual solutions. The overall pattern of the solutions to the problem for general values of the Reynolds number is described. The linear spatial stability of the flows to small steady disturbances is analysed and a few results presented. In particular, details of the solutions and their stability are given asymptotically for small and large values of the Reynolds number. The asymptotic results for the basic flow are linked by direct numerical integration of the flow at several finite positive values of the Reynolds number.

1. Introduction

The structure of vortices, their stability and their breakdown are important in many applications of fluid mechanics, notably to dust devils and tornados in meteorology, and to leading-edge and wing-tip vortices in aerodynamics. Of the extensive literature on vortices, Long's (1958, 1961) papers are a major contribution. He sought a similarity solution to represent a steady rotationally symmetric vortex with axial as well as swirling flow in a uniform incompressible viscous fluid, used the Navier–Stokes equations to derive a system of ordinary differential equations describing the assumed form of solution, and solved the system in the special case of a slightly viscous fluid. Thereby Long (1961) found that there are dual boundary-layer solutions for $M > M_c$ and no solution for $M < M_c$, where the 'flow force' or 'momentum transfer' M is a dimensionless parameter which we shall define explicitly in the next section, and M_c is a critical value which Long calculated approximately. This property of dual solutions is characteristic of a turning point, known so widely now that bifurcation theory is fashionable but less well known when Long found his solutions. This knowledge leads us at once to conclude from the generic case of a turning point that the bifurcation at $M = M_c$ corresponds to the change in the sign of a real eigenvalue of the linearized stability problem of the flow, so that either one of the dual flows is stable and the other unstable, or both are unstable to a different eigenmode. In fact, it has been found by Foster & Duck (1982), Stewartson (1982) and Foster & Smith (1989) that both these flows of an inviscid fluid are temporally unstable to helical modes, i.e. modes which are not rotationally symmetric. Khorrami & Trivedi (1994) have recently extended these results by including the effects of viscosity at large values of the Reynolds number; they also neglected nonparallelism of the basic flow and found new unstable modes.

In view of this fact that the vortices are always unstable in the boundary-layer limit of small viscosity which Long considered, it is surprising that work on Long's problem has been largely confined to this limit. Foster & Jacqmin (1992) have, however, solved the problem asymptotically for large M and fixed values of the Reynolds number R. Shtern & Hussain (1993) also have solved a problem for a vortex closely related to Long's, with the same similarity form of solution, but with a different boundary condition at the axis to drive the vortex, computing the solutions for a wide range of values of the Reynolds number and the flow force. It should be noted, however, that the character of the solution is very different. We shall report results for all values of R, not just large ones, and all values of M. In particular we shall solve the problem asymptotically in the limits of small R and large flow force M, and link our results with those of Long for large R by direct numerical integrations of the system. This will give an overall picture of the results, the occurrence of multiple solutions and their instabilities. We shall mostly confine our attention to rotationally symmetric steady perturbations consistent with the similarity form, although, of course, limitation of the class of perturbations of a basic flow permits a demonstration of its instability but not of its stability.

The phrase 'spatial stability' will often be used in this paper, so it should at once be made clear that this will refer to the spatial development of steady small perturbations. The bifurcation of one steady flow to another is closely related to their stability, so the evolution of small perturbations in space is closely related to their evolution in time, as well as of independent physical importance. However, a steady basic flow might become unstable to oscillatory perturbations at a Hopf bifurcation, and yet the theory of spatial stability to steady perturbations gives no indication of instability. So the restriction to steady perturbations permits the deduction of instability of the basic flow but not stability. Our results for spatial stability also complement those for temporal stability which are cited above.

Benjamin's mechanism of vortex breakdown (cf. Hall 1972; Leibovich 1978), namely the sudden change of one vortex regime to another, depends on the coexistence of two conjugate flows, i.e. equilibria, so that a disturbance may lead to an abrupt change of equilibrium, as in a hydraulic jump or a 'catastrophe'. Other proposed mechanisms of breakdown involve the hydrodynamic instability of a vortex. We shall investigate these possibilities in the context of Long's model, limited though that context is.

2. Formulation of the mathematical problems

2.1. *The basic vortex*

Long (1958) assumed rotationally symmetric steady flow of a uniform incompressible fluid of density ρ, pressure p and kinematic viscosity ν with a Stokes streamfunction $\psi(r,z)$ such that the radial and axial velocity components are

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{2.1}$$

respectively, in terms of cylindrical polar coordinates (r, ϕ, z). He further assumed that there is a flow of similarity form with, essentially,

$$\psi(r,z) = KzF(x), \quad u_\phi(r,z) = KG(x)/r, \quad p(r,z) = -\rho K^2 H(x)/z^2, \tag{2.2}$$

where the similarity independent variable is

$$x = r/z, \tag{2.3}$$

and K is the constant swirl of the vortex at infinity, i.e.

$$ru_\phi \to K \quad \text{as } r \to \infty. \tag{2.4}$$

Another fundamental constant is the dimensionless flow force or momentum transfer defined as

$$M = K^{-2} \int_0^{2\pi} \int_0^{\infty} (u_z^2 + p/\rho)\ r\mathrm{d}r\ \mathrm{d}\phi; \tag{2.5}$$

it can be shown by integrating the z-component of the Navier–Stokes equations over a plane z = constant that

$$M = 2\pi \int_0^{\infty} (F'^2 - x^2 H)/x\ \mathrm{d}x, \tag{2.6}$$

independently of the value of z. Also the Reynolds number may be defined as

$$R = K/\nu. \tag{2.7}$$

This flow, then, is a combined vortex and jet with axis $r = 0$ and velocity components

$$u_r = K(xF' - F)/r, \quad u_\phi = KG/r, \quad u_z = KF'/r, \tag{2.8}$$

where a prime denotes differentiation with respect to x. Substitution into the r-, ϕ-, and z-components of the Navier–Stokes equations now gives

$$x^3(1 + x^2)H' + 3x^4 H = -(F^2 - 2xFF' + G^2), \tag{2.9}$$

$$x(1 + x^2)G'' + (2x^2 - 1)G' = -RFG', \tag{2.10}$$

$$x(1 + x^2)F'' - F' = R(x^3 H - FF'), \tag{2.11}$$

after a little manipulation, integration and elimination. The appropriate boundary conditions determining a smooth solution at $r = 0$ and a flow tending to a uniform swirl K at infinity are

$$F(0) = F'(0) = G(0) = 0, \quad F'(\infty) = 2^{-1/2}, \quad G(\infty) = 1, \quad H(\infty) = 0. \tag{2.12}$$

These equations and conditions determine the steady rotationally symmetric flow, or flows, for given dimensionless parameters R and M. It seems that the problem is well posed, although the ordinary-differential system (2.9)–(2.11) is of fifth order and there are six conditions (2.12), because the rotational symmetry of the system gives F as an even function of r and therefore of x, and the condition $F'(0) = 0$ is redundant if F is a smooth function of x^2 at the origin.

Note (cf. Burggraf & Foster 1977) that M may be determined either by equation (2.6) *after* the solution has been calculated, or by replacing (2.6) by its differential form, namely

$$m' = 2\pi(F'^2 - x^2 H)/x, \quad m(0) = 0, \quad m(\infty) = M. \tag{2.13}$$

The vorticity $\nabla \times \boldsymbol{u}$ and hence the helicity density $\boldsymbol{u} \cdot (\nabla \times \boldsymbol{u})$ can easily be calculated in terms of F, G. It follows that

$$\begin{aligned} J &= \int_0^{2\pi} \int_0^{\infty} \boldsymbol{u} \cdot (\nabla \times \boldsymbol{u})\ r\mathrm{d}r\mathrm{d}\phi \\ &= 2\pi K^2 z^{-1} \int_0^{\infty} [x^{-1}\{(1 + x^2)F' - xF\}G' - x^{-2}\{x(1 + x^2)F'' - F'\}G]\ \mathrm{d}x. \end{aligned} \tag{2.14}$$

On anticipating the results of §3.1, it can be shown that

$$J \to 2^{1/2}\pi K^2 z^{-1} \quad \text{as } R \to 0,$$

and thence that in general J is non-zero. However, the helicity itself is unbounded because J is not integrable from $z = 0$ to $z = \infty$, although the unboundedness does little more than reflect the intrinsic singularity of Long's solution at the origin.

2.2. *The linear spatial stability problem for rotationally symmetric modes*

Burggraf & Foster (1977) considered some small spatial perturbations of Long's boundary-layer flow at large values of the Reynolds number R. We shall generalize their work, revealing the importance of its context by treating not only all values of R but also more-general dependence of the steady perturbations of the flow on the axial coordinate z; this leads, as follows, to the spatial eigenvalue problem governing the decay (or growth) of the flow downstream as it approaches (or leaves, respectively) Long's flow. Write

$$\psi = K(\psi_0 + \psi_1), \quad u_\phi = K(v_0 + v_1)/r, \quad p = \rho K^2(p_0 + p_1), \tag{2.15}$$

where the basic flow is given by

$$\psi_0(r,z) = zF(x), \quad v_0(r,z) = G(x), \quad p_0(r,z) = -H(x)/z^2,$$

as in equation (2.2), and then linearize the Navier–Stokes equations for small perturbations ψ_1, v_1, p_1. It can then be seen that the variables are separable with solutions of the form

$$\psi_1(r,z) = z^\lambda f(x), \quad v_1(r,z) = z^{\lambda-1}g(x), \quad p_1(r,z) = -z^{\lambda-3}h(x), \tag{2.16}$$

where λ is the separation constant, possibly complex. The r-, ϕ- and z-components of the linearized Navier–Stokes equations imply at length that

$$x^3(1+x^2)f''' - x^2[(\lambda-1)+3(\lambda-2)x^2]f'' + (\lambda-1)x[1+3(\lambda-2)x^2]f' - \lambda(\lambda-1)(\lambda-2)x^2 f$$
$$= R[-x^3h' - x^2Ff'' + (\lambda+1)xFf' + (\lambda-3)x^2F'f' - \lambda x^2F''f - \lambda(\lambda-3)xF'f - 2\lambda Ff - 2Gg], \tag{2.17}$$

$$x(1+x^2)g'' - [1+2(\lambda-2)x^2]g' + (\lambda-1)(\lambda-2)xg = R[-\lambda G'f - Fg' + (\lambda-1)F'g], \tag{2.18}$$

$$x^2(1+x^2)f''' - x[1+2(\lambda-2)x^2]f'' + [1+(\lambda-1)(\lambda-2)x^2]f'$$
$$= R[x^4h' - (\lambda-3)x^3h - xFf'' + (\lambda-3)xF'f' + Ff' - \lambda(xF'' - F')f], \tag{2.19}$$

respectively. It helps a little to subtract equation (2.17) from x times equation (2.19) to get

$$(\lambda-2)\{x^2(1+x^2)f'' - x[1+2(\lambda-1)x^2]f' + \lambda(\lambda-1)x^2 f\}$$
$$= R[x^3(1+x^2)h' - (\lambda-3)x^4h - \lambda xFf' + \lambda(\lambda-2)xF'f + 2\lambda Ff + 2Gg]. \tag{2.20}$$

The linearized boundary conditions are that

$$f(0) = f'(0) = g(0) = f'(\infty) = g(\infty) = h(\infty) = 0. \tag{2.21}$$

The perturbation is taken for a fixed parameter M (and R, indeed), so the linearized form of equation (2.5) gives

$$\int_0^\infty (2F'f' - x^2h)/x \, \mathrm{d}x = 0. \tag{2.22}$$

This poses a problem (2.19), (2.18), (2.20)–(2.22) to determine eigenvalues λ and corresponding eigenfunctions f, g, h. We conjecture that there is spatial 'instability', with some steady disturbances growing faster than the basic flow as $z \to \infty$, when there is at least one eigenvalue, of three sequences of eigenvalues whose real parts decrease to $-\infty$, such that $\mathrm{Re}(\lambda) > 1$. (This conjecture is speculative, being based on no more than an analogy with Jeffery–Hamel flows (Banks, Drazin & Zaturska 1988) and some fragmentary evidence below.)

It can be verified that a special solution of equations (2.18)–(2.20) is given in explicit terms of the basic flow by

$$\lambda = 0, \quad f = xF' - F, \quad g = xG', \quad h = xH' + 2H. \tag{2.23}$$

This solution follows because the flow is invariant under the continuous group of translations in the axial direction, $z \mapsto z + \delta$ for all real δ. It is a simple generalization of a result of Burggraf & Foster (1977, §3) for the boundary-layer case. However, it is an eigensolution for all M, R because it in fact also satisfies conditions (2.21) and equation (2.22), i.e. $x^2 H \to \frac{1}{2}$ as $x \to \infty$.

2.3. *The spatial stability problem for asymmetric modes*

Next we examine stability to helical modes. Both variables ϕ, z may be separated by assuming that the perturbations are proportional to $e^{in\phi}$ so that $v_1(r,\phi,z) = z^{\lambda-1}e^{in\phi}\hat{v}(x)$ etc. In this way the linear stability of the flow to spatial modes may be treated without recourse to solving a partial differential system. Thus we express

$$(u_r, u_\phi, u_z) = K(u_0 + u_1, v_0 + v_1, w_0 + w_1)/r, \quad p = \rho K^2(p_0 + p_1), \tag{2.24}$$

where $(u_0, v_0, w_0) = (xF' - F, G, F')$, $p_0 = -H/z^2$, as in equation (2.2). Then, on linearizing the Navier–Stokes equations for small perturbations u_1, v_1, w_1, p_1, it can be seen that the variables may be separated by taking spatial modes of the form

$$(u_1, v_1, w_1) = z^{\lambda-1}e^{in\phi}(\hat{u}(x), \hat{v}(x), \hat{w}(x)), \quad p_1 = -z^{\lambda-3}e^{in\phi}\hat{p}(x), \tag{2.25}$$

where the non-negative integer n is the azimuthal wavenumber and the complex eigenvalue λ gives the rate of spatial growth or decay in the axial direction.

It follows at length that the linearized r-equation gives

$$\mathscr{L}_n\hat{u} - 2in\hat{v} = R[-x^3\hat{p}' - xF\hat{u}' + x^2F''\hat{u} + (\lambda-3)xF'\hat{u} + 2F\hat{u} + inG\hat{u} - 2G\hat{v} - x^3F''\hat{w}], \tag{2.26}$$

the ϕ-equation gives

$$2in\hat{u} + \mathscr{L}_n\hat{v} = R[-inx^2\hat{p} + xG'\hat{u} - xF\hat{v}' + (\lambda - 1)xF'\hat{v} + inG\hat{v} - x^2G'\hat{w}], \tag{2.27}$$

the z-equation gives

$$\mathscr{L}_n\hat{w} + \hat{w} = R[x^4\hat{p}' - (\lambda-3)x^3\hat{p} + (xF'' - F')\hat{u} - xF\hat{w}' - x^2F''\hat{w} + (\lambda-2)xF'\hat{w} + F\hat{w} + inG\hat{w}], \tag{2.28}$$

and the continuity equation gives

$$x\hat{u}' + in\hat{v} - x^2\hat{w}' + (\lambda - 1)x\hat{w} = 0, \tag{2.29}$$

where the linear operator $\mathscr{L}_n$ is defined by

$$\mathscr{L}_n = x^2(1 + x^2)\mathrm{d}^2/\mathrm{d}x^2 - x[1 + 2(\lambda - 2)x^2]\mathrm{d}/\mathrm{d}x + [-n^2 + (\lambda - 1)(\lambda - 2)x^2]. \tag{2.30}$$

The boundary conditions (cf. Batchelor & Gill 1962, p. 534) are that

$$\hat{w}(0) = \hat{p}(0) = \hat{u}(\infty) = \hat{v}(\infty) = \hat{w}(\infty) = \hat{p}(\infty) = 0, \tag{2.31}$$

and

$$\hat{u}(0) = \hat{v}(0) = 0 \text{ if } n \neq 1 \text{ or } \hat{v}(0) = i\hat{u}(0) \text{ if } n = 1. \tag{2.32}$$

This is the spatial eigenvalue problem for general asymmetric perturbations. Note that linearization of equation (2.5) for fixed M is automatically satisfied if $n \neq 0$.

It can be verified that a special solution of equations (2.26)–(2.29) is

$$\left.\begin{aligned} &\lambda = 0,\ n = 1,\ \hat{p} = H',\ \hat{u} = [(x^2F'' - xF' + F) - \mathrm{i}G]/x, \\ &\hat{v} = [(xG' - G) + \mathrm{i}(xF' - F)]/x,\ \hat{w} = (xF'' - F')/x. \end{aligned}\right\} \quad (2.33)$$

This solution follows because the basic flow is invariant under the continuous group of translations perpendicular to the axis. It is an eigensolution for all M, R because it also satisfies the boundary conditions.

Finally, we note that the temporal linear stability problem is reducible to an eigenvalue problem for normal modes with an inseparable partial differential system with independent variables r, z by assuming that perturbations are linear combinations of modes of the form $v_1(r, \phi, z, t) = \mathrm{e}^{\mathrm{i}(st+n\phi)}\hat{v}(r, z)$ etc.

3. The asymptotic solution for small R

In this section we shall find the basic solution F, G, H and solve its spatial stability problem asymptotically in the Stokes limit.

3.1. *The basic vortex*

First find the basic flow by assuming that there exist convergent power series of the form

$$F = F_0 + RF_1 + \dots,\quad G = G_0 + RG_1 + \dots,\quad H = H_{-1}/R + H_0 + RH_1 + \dots\ \text{as}\ R \to 0. \quad (3.1)$$

We take the leading term in the expansion of the pressure function H to be of order R^{-1} to balance the pressure with the viscous forces and neglect the inertia in the leading approximation. Therefore equations (2.9)–(2.11) give

$$x^3(1 + x^2)H'_{-1} + 3x^4H_{-1} = 0, \quad (3.2)$$

$$x(1 + x^2)G''_0 + (2x^2 - 1)G'_0 = 0, \quad (3.3)$$

$$x(1 + x^2)F''_0 - F'_0 = x^3H_{-1}, \quad (3.4)$$

respectively. Solving these equations and the boundary conditions (2.12), we find at length that the basic flow in this limit is given by

$$F_0(x) = 2^{-1/2}[(1 + x^2)^{1/2} - 1] - \tfrac{1}{2}\alpha[1 - (1 + x^2)^{-1/2}],\quad G_0(x) = 1 - (1 + x^2)^{-1/2},$$

$$H_{-1}(x) = \alpha(1 + x^2)^{-3/2}, \quad (3.5)$$

where α is an arbitrary constant. However, we must satisfy equation (2.6) in order that the flow force has its given value. To be consistent with our Stokes approximation this implies that

$$M = M_{-1}/R + M_0 + RM_1 + \dots\ \text{as}\ R \to 0 \quad (3.6)$$

for some constants $M_{-1}, M_0, M_1, \dots$. Then equation (2.6) gives M_{-1} in terms of H_{-1} independently of F_0, H_0, and it follows that $M_{-1} = -2\pi\alpha$. It can at great length be shown that G_1 is a function which is linear in α and introduces no new constant, but H_0 is a function which is quadratic in α and introduces a new constant, β say, on integration; some of the details are in Appendix A (which is not printed with this paper, although a copy may be obtained on application to any author or to the Editor of the Journal of Fluid Mechanics). We have also evaluated M_0 on use of the forms for F_0, H_0, finding that

$$M_0 = 3\pi(\log 2 - 1) - 2^{-1/2}\pi\alpha(\log 2 + 1) + \tfrac{1}{8}\alpha^2\pi(9 - 8\log 2) - 2^{-1/2}\pi\beta. \quad (3.7)$$

At this stage we may take the limit as $R \to 0$ either (i) for fixed $MR = M_{-1}$, in which case $M_0 = M_1 = ... = 0$, we deduce that $\alpha = -M_{-1}/2\pi$, find β from equation (3.7) with $M_0 = 0$, and proceed as far with the calculation as is required; or (ii) for fixed $M = M_0$, in which case $M_{-1} = M_1 = ... = 0$, $\alpha = 0$, and $\beta = 2^{1/2}[3(\log 2 - 1) - M_0/\pi]$.

3.2. *The linear spatial stability of rotationally symmetric modes*

Next let us solve the spatial eigenvalue problem for rotationally symmetric modes. We may expand

$$\lambda = \lambda_0 + R\lambda_1 + ..., \quad f = f_0 + Rf_1 + ..., \quad g = g_0 + Rg_1 + ..., \quad h = h_{-1}/R + h_0 + ... \text{ as } R \to 0, \tag{3.8}$$

and substitute the expansions into the problem (2.18)–(2.21). First note that the problem is independent of the basic flow in the Stokes limit. Also boundary conditions (2.21) give

$$f_0(0) = f_0'(0) = g_0(0) = f_0'(\infty) = g_0(\infty) = h_{-1}(\infty) = 0. \tag{3.9}$$

It can be seen that equation (2.18) gives the g_0-equation,

$$x(1+x^2)g_0'' - [1 + 2(\lambda_0 - 2)x^2]g_0' + (\lambda_0 - 1)(\lambda_0 - 2)xg_0 = 0, \tag{3.10}$$

which decouples from the other two equations. This can be shown to be a hypergeometric equation for g_0 as a function of $1+x^2$ with $a = -\frac{1}{2}(\lambda_0 - 2), b = -\frac{1}{2}(\lambda_0 - 1), c = -(\lambda_0 - \frac{5}{2})$ in standard notation (cf. Abramowitz & Stegun 1964, Chap. 15). It follows at length that the eigensolution then is

$$\lambda_0 = -l, \quad f_0(x) = 0, \quad g_0(x) = Q_l(x), \quad h_{-1}(x) = 0, \tag{3.11}$$

for $l = 0, 1, 2, ...,$ where

$$\left.\begin{aligned} Q_{2j}(x) &= x^2(1+x^2)^{-2j-3/2}\sum_{k=0}^{j}\frac{(-j)_k(-j+\frac{1}{2})_k}{(-2j-\frac{1}{2})_k}\frac{(1+x^2)^k}{k!}, \\ Q_{2j+1}(x) &= x^2(1+x^2)^{-2j-5/2}\sum_{k=0}^{j}\frac{(-j)_k(-j-\frac{1}{2})_k}{(-2j-\frac{3}{2})_k}\frac{(1+x^2)^k}{k!} \end{aligned}\right\} \tag{3.12}$$

for $j = 0, 1, ...,$ and the Pochhammer symbols are defined by $(q)_k = \Gamma(q+k)/\Gamma(q)$ for all q. For these swirling modes higher terms f_n, h_n are not all zero. Note also that with $\lambda_0 = 0$ we recover a Stokes limit $g_0 = xG_0'$ of the eigensolution given, for all R, by equations (2.23).

In addition, there are eigensolutions for which g_0 is, but f_0, h_{-1} are not, zero everywhere. Equations (2.19), (2.20) give

$$x^2(1+x^2)f_0''' - x[1+2(\lambda_0-2)x^2]f_0'' + [1+(\lambda_0-1)(\lambda_0-2)x^2]f_0' = x^4h_{-1}' - (\lambda_0-3)x^3h_{-1}, \tag{3.13}$$

$$(\lambda_0-2)\{x(1+x^2)f_0'' - [1+2(\lambda_0-1)x^2]f_0' + \lambda_0(\lambda_0-1)xf_0\} = x^2(1+x^2)h_{-1}' - (\lambda_0-3)x^3h_{-1}. \tag{3.14}$$

It can be shown by use of a little calculus and algebra that if $h_{-1}(x) = 0$ for all x and

$$x(1+x^2)f_0'' - [1 + 2(\lambda_0 - 1)x^2]f_0' + \lambda_0(\lambda_0 - 1)xf_0 = 0 \tag{3.15}$$

then *both* equations (3.13), (3.14) are satisfied. Now equation (3.15) is the same as (3.10) on replacing λ_0 by $\lambda_0 + 1$, and so is another hypergeometric equation for f_0

as a function of $1+x^2$. Thus, using the boundary conditions $f_0(0) = f_0'(\infty) = 0$, we deduce that

$$\left.\begin{aligned} \lambda_0 &= 0, \ f_0(x) = 1-(1+x^2)^{-1/2}, \ g_0(x) = h_{-1}(x) = 0, \\ \lambda_0 &= -l-1, \ f_0(x) = Q_l(x), \ g_0(x) = h_{-1}(x) = 0 \end{aligned}\right\} \tag{3.16}$$

for $l = 0, 1, 2, \ldots$. Note that, on using basic flow (3.5) with $\alpha = 0$ and equations (2.23) in the Stokes limit, we verify the solution above for $\lambda_0 = 0$ with $f_0 = xF_0' - F_0, h_{-1} = xH_{-1}' + 2H_{-1}$.

Also the derivative of the quotient of equation (3.14) and x gives

$$x(1+x^2)h_{-1}'' + [1-2(\lambda_0-4)x^2]h_{-1}' + (\lambda_0-3)(\lambda_0-4)xh_{-1} = 0, \tag{3.17}$$

on elimination of f_0 by use of equation (3.13). This is again a hypergeometric equation for h_{-1} as a function of $1+x^2$, but now with $a = -\frac{1}{2}(\lambda_0-4), b = -\frac{1}{2}(\lambda_0-3), c = -(\lambda_0-\frac{7}{2})$. From this equation we can find h_{-1}, using its boundary condition at infinity, find f_0 from equation (3.14), and apply the two boundary conditions on f_0. This eventually gives

$$\lambda_0 = -l, \ f_0(x) = -\tfrac{1}{2}Q_l(x), \ g_0(x) = 0, \ h_{-1}(x) = x^{-1}Q_l'(x). \tag{3.18}$$

Note that this solution is not unique, because for each eigenvalue λ_0 we may add to the solution an arbitrary multiple of the solution (3.16). It may thereby be shown to be compatible with the solution for $\lambda_0 = 0$ which follows on using basic flows (3.5) with $\alpha = \infty$ and equations (2.23) in the Stokes limit.

3.3. *The linear spatial stability of asymmetric modes*

We next formulate the eigenvalue problem for asymmetric modes, expanding

$$\left.\begin{aligned} \lambda &= \lambda_0 + R\lambda_1 + \ldots, \ \hat{u} = \hat{u}_0 + R\hat{u}_1 + \ldots, \ \hat{v} = \hat{v}_0 + R\hat{v}_1 + \ldots, \\ \hat{w} &= \hat{w}_0 + R\hat{w}_1 + \ldots, \ \hat{p} = \hat{p}_{-1}/R + \hat{p}_0 + \ldots, \ \text{as } R \to 0, \end{aligned}\right\} \tag{3.19}$$

and finding

$$\mathscr{L}_n\hat{u}_0 - 2in\hat{v}_0 = -x^3\hat{p}_{-1}', \tag{3.20}$$

$$2in\hat{u}_0 + \mathscr{L}_n\hat{v}_0 = -inx^2\hat{p}_{-1}, \tag{3.21}$$

$$\mathscr{L}_n\hat{w}_0 + \hat{w}_0 = x^4\hat{p}_{-1}' - (\lambda_0-3)x^3\hat{p}_{-1}, \tag{3.22}$$

$$x\hat{u}_0' + in\hat{v}_0 - x^2\hat{w}_0' + (\lambda_0-1)x\hat{w}_0 = 0. \tag{3.23}$$

The boundary conditions (2.31), (2.32) still apply in the Stokes limit.

The eigensolution (2.33) in the Stokes limit gives two eigensolutions here for $\lambda_0 = 0, n = 1$, namely

$$\left.\begin{aligned} \hat{u}_0(x) &= \{2^{-1/2}[(1+2x^2)/(1+x^2)^{3/2} - 1] - i[1 - 1/(1+x^2)^{1/2}]\}/x, \\ \hat{v}_0(x) &= \{(1+2x^2)/(1+x^2)^{3/2} - 1 + 2^{-1/2}i[1-1/(1+x^2)^{1/2}]\}/x, \\ \hat{w}_0(x) &= -x^2/2^{1/2}(1+x^2)^{3/2}, \ \hat{p}_{-1}(x) = 0, \end{aligned}\right\} \tag{3.24}$$

and

$$\left.\begin{aligned} \hat{u}_0(x) &= [(1+2x^2+4x^4)/(1+x^2)^{5/2} - 1]/2x, \\ \hat{v}_0(x) &= i[1-(1+2x^2)/(1+x^2)^{3/2}]/2x, \ \hat{w}_0(x) = 3x^2/2(1+x^2)^{5/2}, \\ \hat{p}_{-1}(x) &= -3x/(1+x^2)^{5/2}. \end{aligned}\right\} \tag{3.25}$$

Each of the eigensolutions in this section gives spatial stability, as we might have anticipated – after all, they are no more than some solutions of the linear Stokes equations in the disguise of an unfamiliar system of coordinates. However, the basic flows and their stability characteristics serve not only to verify the stability of the basic flow in the Stokes approximation, but also to provide a contrast with Long's boundary-layer solutions of the next section, and to use as a predictor to start off the numerical solutions.

4. The asymptotic solution for large R

4.1. *The boundary-layer limit of the basic vortex*

After deriving equations (2.9)–(2.11), Long (1961) went on to transform the variables to

$$y = Rx/2^{1/2} = Kr/\sqrt{2}\nu z, \quad f(y) = RF(x), \quad \Gamma(y) = G(x), \quad s(y) = H(x)/R^2, \tag{4.1}$$

took the limit as $R \to \infty$ for fixed M, and deduced the 'boundary-layer' equations

$$2y^3 s' + \Gamma^2 = 0, \tag{4.2}$$

$$y\Gamma'' - (1-f)\Gamma' = 0, \tag{4.3}$$

$$yf'' - (1-f)f' - 4y^3 s = 0, \tag{4.4}$$

where here a prime denotes differentiation with respect to y. (These equations may alternatively be obtained by starting from the boundary-layer equations derived directly from the Navier–Stokes equations: such boundary-layer equations have been called the quasi-cylindrical equations by Hall (1966). Note also that this function f is different from the one defined in equation (2.16).) In this case,

$$u_r = \nu(yf' - f)/r, \quad u_\phi = K\Gamma/r, \quad u_z = Kf'/2^{1/2}r, \quad p = -\rho K^4 s/\nu^2 z^2.$$

Similarly, equation (2.13) gives

$$m' = \pi(f'^2 - 4y^2 s)/y. \tag{4.5}$$

The appropriate conditions for this boundary-layer form are that

$$f(0) = f'(0) = \Gamma(0) = m(0) = 0, \quad f'(\infty) = \Gamma(\infty) = 1, \quad s(\infty) = 0, \quad m(\infty) = M. \tag{4.6}$$

Long calculated the solutions of this problem numerically. His results are so fundamental to our work that we reproduce a few of them briefly in the bifurcation diagram of figure 1: we have, following Long, chosen $[f'/2y]_{y=0}$ as the state variable and M as the behaviour variable but we have used our own numerical results (see §5), which are consistent with those of Burggraf & Foster (1977, Table 1). It can be seen that there are two solutions for $M > M_c$ and none for $M < M_c$, where in fact $M_c = 3.75$. One solution, said to be of type II, has an axial velocity profile with a maximum on an annulus, one minimum on the axis $r = 0$ and another at $r = \infty$; the other solution, said to be of type I, has however, for $M > 4.71$, an axial profile of jet-like shape with a single maximum on the axis $r = 0$ and minimum at $r = \infty$. The solutions of types I and II join at the bifurcation point where $M = M_c$ – which we anticipate is a point of marginal stability with respect to one mode.

4.2. *The linear spatial stability of rotationally symmetric modes*

The boundary-layer limit of the spatial stability problem for rotationally symmetric modes may be formulated by a procedure similar to the one given in the previous

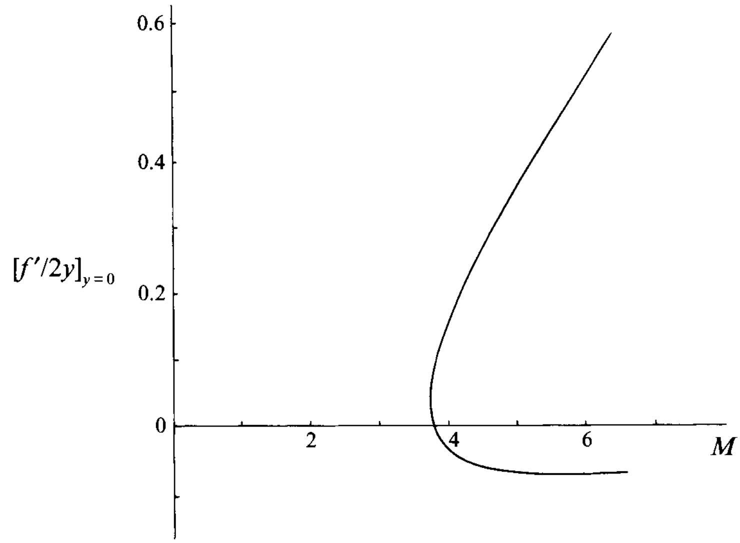

FIGURE 1. The bifurcation diagram in the $(M, [f'/2y]_{y=0})$-plane for the boundary-layer solutions of Long (1961). Note that $[vzu_z/K^2]_{r=0} = [f'/2y]_{y=0} = [F'/Rx]_{x=0}$.

subsection. We define

$$\hat{f}(y) = Rf(x), \quad \hat{\Gamma}(y) = g(x), \quad \hat{s}(y) = h(x)/R^2,$$

substitute into equations (2.17)–(2.19) and take the limit as $R \to \infty$. This gives

$$y^3\hat{s}' + \Gamma\hat{\Gamma} = 0, \tag{4.7}$$

$$y\hat{\Gamma}'' - (1-f)\hat{\Gamma}' - (\lambda - 1)f'\hat{\Gamma} + \lambda\Gamma'\hat{f} = 0, \tag{4.8}$$

$$y^2\hat{f}''' - y(1-f)\hat{f}'' + [1-f-(\lambda-3)yf'']\hat{f}' + \lambda(yf''-f')\hat{f} - 4[y^4\hat{s}' - (\lambda-3)y^3\hat{s}] = 0, \tag{4.9}$$

for the determination of the eigenvalue λ to leading order. Similarly

$$\hat{m}' = 2\pi(f'\hat{f}' - 2y^2\hat{s})/y. \tag{4.10}$$

The linearized boundary conditions give

$$\hat{f}(0) = \hat{f}'(0) = \hat{\Gamma}(0) = \hat{m}(0) = \hat{f}'(\infty) = \hat{\Gamma}(\infty) = \hat{s}(\infty) = \hat{m}(\infty) = 0. \tag{4.11}$$

The eigensolution (2.23) for all R, M becomes

$$\lambda = 0, \quad \hat{f} = yf' - f, \quad \hat{\Gamma} = y\Gamma', \quad \hat{s} = ys' + 2s$$

in the boundary-layer limit. Burggraf & Foster (1977, §3) originally found this solution for the boundary-layer limit, and also found numerically a second solution with $\lambda = 0, \hat{\Gamma} = \hat{s} = 0$: we note the consistency of the second solution with the analysis of §3.2 for small R.

4.3. *The linear spatial stability of asymmetric modes*

A balance of terms in the spatial stability problem for asymmetric modes is possible in the limit as $R \to \infty$ if

$$\hat{u}(x) = \hat{u}_0(y) + ..., \quad \hat{v}(x) = \hat{v}_0(y) + ..., \quad \hat{w}(x) = \hat{w}_0(y) + ..., \quad \hat{p}(x) = R^2\hat{p}_0(y) + ... \tag{4.12}$$

On substituting these forms into equations (2.26)–(2.29) and taking the boundary-layer limit, we obtain the equations

$$2\Gamma\hat{v}_0 - in\Gamma\hat{u}_0 + 2y^3\hat{p}_0' = 0, \tag{4.13}$$

$$-y\Gamma'\hat{u}_0 - in\Gamma\hat{v}_0 + 2iny^2\hat{p}_0 = 0, \tag{4.14}$$

$$2^{-1/2}(yf'' - f')\hat{u}_0 + in\Gamma\hat{w}_0 = 0, \tag{4.15}$$

$$y\hat{u}_0' + in\hat{v}_0 = 0. \tag{4.16}$$

Note that these equations are of inviscid form and are independent of λ . We conjecture that λ is determined at the next stage of the approximation for large R. Note further that we can eliminate $\hat{p}_0$ and $\hat{v}_0$ to get an equation for $\hat{u}_0$ in terms of Γ: we find that

$$y^2\Gamma\hat{u}_0'' + y\Gamma\hat{u}_0' - (y^2\Gamma'' - y\Gamma' + n^2\Gamma)\hat{u}_0 = 0. \tag{4.17}$$

We further conjecture that this 'outer' solution is common to all eigensolutions for $n = 1, 2, ...$ but that appropriate 'inner' solutions will determine the limits of the eigenvalues λ.

These conjectures may be supported by appeal to the special eigensolution (2.33). On abstracting from $\hat{u}$ the leading-order term, which is proportional to Γ/y, it is easy to see that when $n = 1$ the equation for $\hat{u}_0$ is satisfied identically. It is then possible to deduce $\hat{w}_0, \hat{p}_0$ from the above equations, or, indeed, directly from the terms of leading order in equation (2.33).

4.4. *Perturbation of a basic vortex*

Next we consider perturbations of a given boundary-layer solution of Long's vortex due to small changes of the flow force in order, especially, to elucidate the structure of the turning point at $M = M_c$.

The linear stability problem of §4.2 may be summarized as giving

$$\psi = \nu[zf(y) + \psi_1], \; \psi_1(r, z) = zA(z)\hat{f}(y), \tag{4.18}$$

where

$$z\frac{\mathrm{d}A}{\mathrm{d}z} = (\lambda - 1)A \tag{4.19}$$

for fixed M. Now if $M = M_c$ we find that the principal or highest eigenvalue $\lambda = 1$. Further, if $M = M_c + \epsilon$ then

$$\lambda - 1 = \epsilon^{1/2}\lambda_{1/2} + \epsilon\lambda_1 + \cdots \quad \text{as} \quad \epsilon \downarrow 0. \tag{4.20}$$

This is the linear theory of the spatial development of the least stable axisymmetric disturbances for fixed M close to M_c.

Also it may be shown by perturbation theory that the basic similarity solution is given by

$$\psi = \nu[zf_0(y) + \psi_1], \; \psi_1(r, z) = zA(z)u_1(y) \tag{4.21}$$

where

$$0 = \epsilon a^2 - A^2 \tag{4.22}$$

approximately, and a^2 can be expressed explicitly as the quotient of a linear function of the adjoint eigensolution evaluated at the boundaries $y = 0, \infty$ and of an integral of the product of the adjoint eigensolution and a quadratic function of the eigensolution. (The details of the perturbation are given in Appendix B, a copy of which may be obtained on application to any author or to the Editor of the Journal of Fluid Mechanics.)

It follows that the weakly nonlinear theory for spatial development of the flows with small changes of M from M_c gives equation (4.21) where (4.19) and (4.22) are synthesized as the equation

$$z\frac{\mathrm{d}A}{\mathrm{d}z} = \frac{1}{2}a^{-1}\lambda_{1/2}(A^2 - \epsilon a^2) \tag{4.23}$$

governing the growth downstream of the slowly varying small amplitude A. This equation may be derived directly by perturbation theory for small ϵ and A. It governs the initial stages of the abrupt spatial development of disturbances of the basic flow for $M = M_c$ when M is slightly less than M_c as well as the spatial stability and instability of both the dual flows when M is slightly greater than M_c. The quantitative behaviour depends on the eigensolution at $M = M_c$.

5. Numerical solutions

5.1. *Numerical methods*

We have indicated that when the Reynolds number R is large there are two solutions of Long's problem for each $M > M_c$, of which the type I solution is stable and the type II is unstable to disturbances of form (2.16) (but both are unstable to inviscid helical disturbances). However, when R is small there is a unique solution for given M and it is stable. This poses the problem of finding the number and behaviour of the solutions for intermediate values of R and all M. This bifurcation problem will be addressed next by numerical methods.

To find the basic flow we solve the system (2.9)–(2.11), (2.13), (2.12). However, because the equations are singular at $x = 0$, we use power series for F, G, H, m to integrate away from the origin. We find that the series have the forms

$$\left.\begin{aligned} F(x) &= ax^2 + a_4x^4 + ..., \quad G(x) = bx^2 + b_4x^4 + ..., \quad RH(x) = c + c_2x^2 + ..., \\ m(x) &= m_2x^2 + m_4x^4 + ..., \end{aligned}\right\} \tag{5.1}$$

where $m_2 = \pi(4a^2 - c/R), m_4 = \pi(2aa_4 - c_2/4R), a_4 = \frac{1}{8}(c - 2a^2R - 2a), b_4 = \frac{1}{4}b(3 + aR), c_2 = -\frac{1}{2}[3c + R(b^2 - 3a^2)]$ and a, b, c are constants. Further, for $x \gg 1$ we use the asymptotic results

$$F'(x) = 2^{-1/2} + O(x^{-2}), \quad G(x) = 1 + O(x^{-1}), \quad H(x) = \tfrac{1}{2}x^{-2} + O(x^{-3}) \quad \text{as} \quad x \to \infty. \tag{5.2}$$

Large ranges of integration are needed because of the algebraic rather than exponential decay.

It is possible in principle to use these forms to integrate equations (2.9)–(2.11), (2.13) by fixing, say, R, a and then finding b, c iteratively by requiring that $F'(x) \to 2^{-1/2}, G(x) \to 1$ as $x \to \infty$. We see from relations (5.2) that the latter limit can be replaced by $H(x) \sim G(x)/2x^2$. The integration gives rise to the second parameter, $M = m(\infty)$. Of course, the iterative procedure to find b, c is based on the assumption that a solution exists. It is also possible to fix R, M and determine a, b, c iteratively by requiring that $F'(x) \to 2^{-1/2}, G(x) \to 1$ as $x \to \infty$. We have used this procedure for $R \leqslant 15$ to obtain solutions, and we present some details below. However, Long (1961) devised a method of rescaling the variables in his numerical solution of the boundary-layer problem in order to reduce the number of unknowns. We have extended the method to apply it to the problem for finite R by writing

$$F(x) = R^{-1}\bar{x}\bar{F}(\bar{x}), \quad G(x) = (c/R^3)^{1/4}\bar{G}(\bar{x}), \quad H(x) = (c/R)\bar{H}(\bar{x}), \quad m(x) = (c/R^3)^{1/2}\bar{m}(\bar{x}), \tag{5.3}$$

where $\bar{x} = \bar{R}x, \bar{R} = (Rc)^{1/4}$. As usual, we assume a fixed pair of values of M, R. The governing equations (2.9)–(2.11), (2.13) now become

$$\bar{x}^3(1+\bar{x}^2/\bar{R}^2)\bar{H}' + 3\bar{x}^4\bar{H}/\bar{R}^2 = -\bar{G}^2 + \bar{x}^2[2\bar{F}(\bar{x}\bar{F}'+\bar{F}) - \bar{F}^2]/\bar{R}^2, \tag{5.4}$$

$$\bar{x}(1+\bar{x}^2/\bar{R}^2)\bar{G}'' + (2\bar{x}^2/\bar{R}^2 - 1 + \bar{x}\bar{F})\bar{G}' = 0, \tag{5.5}$$

$$\bar{x}(1+\bar{x}^2/\bar{R}^2)(\bar{x}\bar{F}'' + 2\bar{F}') + (\bar{x}\bar{F} - 1)(\bar{x}\bar{F}' + \bar{F}) = 2^{1/2}\bar{x}^3\bar{H}, \tag{5.6}$$

$$\bar{m}' = 2\pi[(\bar{x}\bar{F}'+\bar{F})^2 - \bar{x}^2\bar{H}]/\bar{x}. \tag{5.7}$$

For $0 < \bar{x} \ll 1$ we find that $\bar{F}(\bar{x}) = \bar{a}\bar{x} + \bar{a}_3\bar{x}^3$, $\bar{G}(\bar{x}) = \bar{b}\bar{x}^2 + \bar{b}_4\bar{x}^4$, $\bar{H}(\bar{x}) = 1 + \bar{c}_2\bar{x}^2, \bar{m}(\bar{x}) = \pi(4a^2-1)\bar{x}^2 + \bar{m}_4\bar{x}^4$ approximately, where $\bar{a} = a(R/c)^{1/2}, \bar{b} = b(R/c^3)^{1/4}$ and $\bar{a}_3, \bar{b}_4, \bar{c}_2, \bar{m}_4$ are functions of $\bar{a}, \bar{b}, R$. Also we require that $\bar{F} \to 2^{-1/2}(R^3/c)^{1/4}$, $\bar{G} \to (R^3/c)^{1/4}$, $\bar{H} \sim (R^3/c)^{1/2}/2\bar{x}^2$ for $\bar{x} \gg 1$. So we can, for example, assign values to $\bar{R}, \bar{a}$ and determine $\bar{b}$ by requiring that

$$\bar{H} - \bar{G}^2/2\bar{x}^2 \to 0 \text{ as } \bar{x} \to \infty, \tag{5.8}$$

in a way analogous to Long's. Then, from the limiting value of $\bar{G}(\bar{x})$ as $\bar{x} \to \infty$ and the definition of $\bar{R}$ we can determine $R, c, a, b, m(\infty)$ in turn. As noted earlier, given a value of $\bar{R}$ we can determine $\bar{a}, \bar{b}$ if we not only satisfy condition (5.8) but also use a prescribed value of $m(\infty) = M$. Note also that on taking the limit as $\bar{R} \to \infty$ we recover a form equivalent to Long's.

5.2. *Numerical results*

Using these methods, we first calculated solutions for small values of the Reynolds number R (checking each by use of the Stokes solutions of §3) and then increased R bit by bit. We took $M = 4, > M_c$, so that $\alpha = 0$ in solution (3.5), and found solutions, starting at $R = 0.1$, for increasing values of R. The solutions approached the boundary-layer solution of type I as R became large; this is demonstrated by the plot of a scaled axial velocity (w for $0 \leqslant R \leqslant 1$ and w/R for $1 \leqslant R$) versus R in figure 2(*a*), where $w = [zu_z/K]_{r=0} = [F'/x]_{x=0}$. We have repeated these calculations with $M = 6$ and have also plotted the results in figure 2(*a*): note again the continuation from the Stokes analytic result to Long's boundary-layer result for the solution of type I. Some scaled axial velocity profiles are shown in figure 3. In figure 3(*a*) we plot, for $M = 4$, w/R versus x for $R = 1, 5, 10$, and in figure 3(*b*) we plot w/R versus Rx (the boundary-layer variable) for $R = 10, 15, 80.244$. (The precise choice of the value 80.244 was merely a product of the numerical method.) Note that the value of R at which the axial velocity develops an off-axis maximum lies between 5 and 10; no special calculations were performed to determine this value more accurately.

We have also obtained solutions starting with the boundary-layer solution of type II for $M = 4$ and a large value of R: the results were then calculated for progressively smaller values of R, and are summarized in figure 2(*b*). The results are highly suggestive of a turning point at $R = 10.8$. However, the computations were proving difficult and we did not proceed further. Indeed, it appeared that as the ends of the curves shown in figure 2(*b*) are approached the coefficient $\bar{b}$ in $\bar{G}(\bar{x})$ tends to zero. Since this implies that $\bar{b}_4$ tends to zero, it appears that the profile for $\bar{G}(\bar{x})$ (and hence for $G(x)$) may eventually develop a thin shear layer in which its variation between 0

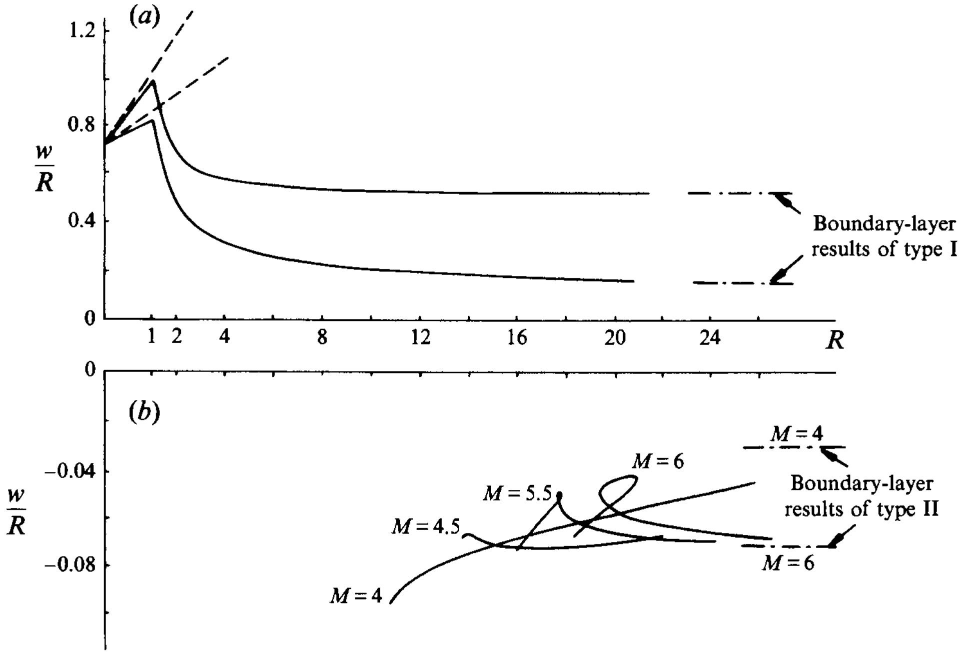

FIGURE 2. Plots of w/R versus R for various values of M, where $w = [zu_z/K]_{r=0} = [F'/x]_{x=0}$. (*a*) The continuation of the solution of type I. The dashed curves show the corresponding small-R results from the first two terms of the Stokes solution. Note that w is plotted for $R < 1$. (*b*) The continuation of the solution of type II.

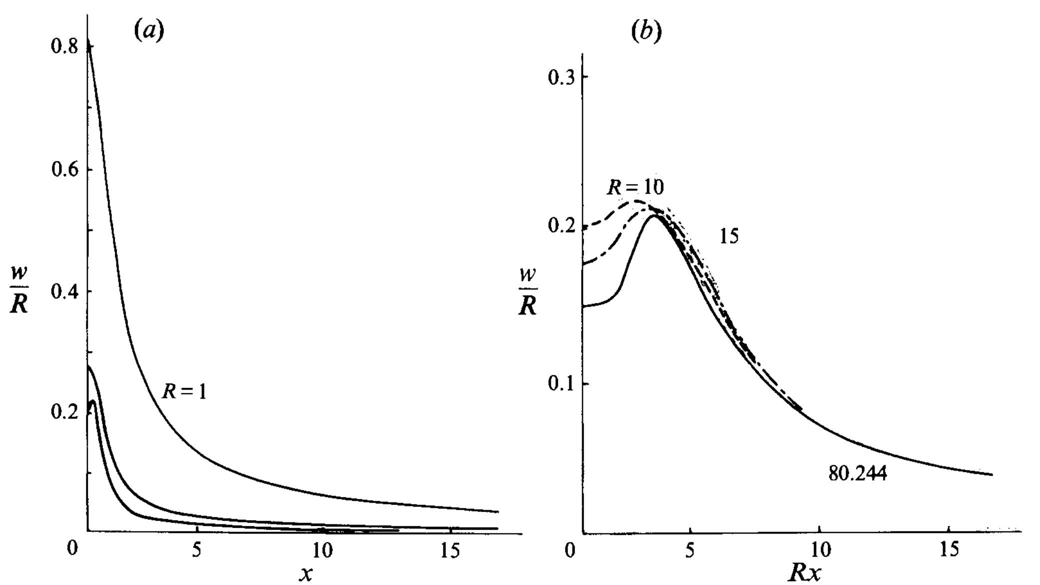

FIGURE 3. Scaled axial velocity profiles, $w = [zu_z/K]_{r=0}$, for $M = 4$. (*a*) Plots of w/R versus x for various values of $R \leqslant 10$. (*b*) Plots of w/R versus Rx for various values of $R \geqslant 10$.

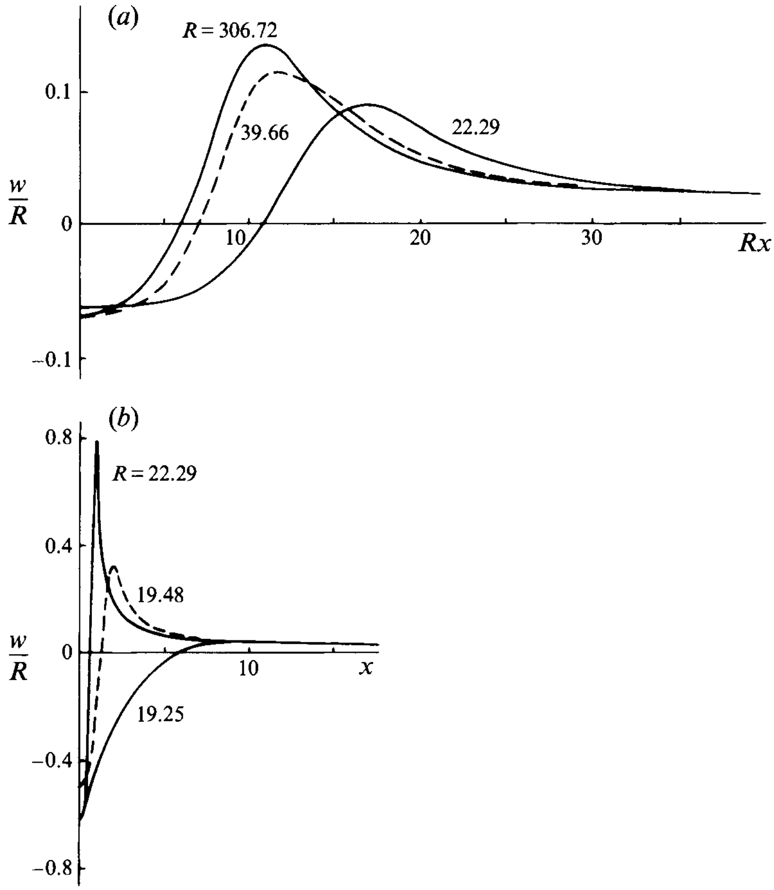

FIGURE 4. Scaled axial velocity profiles for $M = 6$. (*a*) Plots of w/R versus Rx for various values of $R \geqslant 22.29$. (*b*) Plots of w/R versus x for various values of $R \leqslant 22.29$.

and 1 takes place away from the origin (see Foster & Smith 1989, §3.2), who found such a shear layer for $M \gg 1$). We again repeated the calculations for $M = 6$, and the results are summarized in figure 2(*b*). The results for $M = 4$ and $M = 6$ are strikingly different: note that with $M = 6$ there are two turning points which result in at least four solutions for $19.45 < R < 20.90$. (In addition to the four solutions shown in figure 2(*a*, *b*), Dr V. Shtern (private communication) has reported turning points on some curves with fixed M such that they return to the right as R increases.) As in the case when $M = 4$, with $R \approx 18.5$ the calculations were proving difficult and we did not proceed further along the solution curve. However, to help with the interpretation of the solution surface in $(R, M, w/R)$-space, we obtained results for other values of M: in particular, we present results when $M = 5.5$. Note that the region of multiple solutions has greatly diminished ($17.7 < R < 17.9$), and we anticipate that a cusp occurs at $M = M^*, R = R^*$, where $M^* \approx 5.3, R^* \approx 17$. Results for $M = 4.5$ also are plotted. There are two different flows at the point where the solution curve for $M = 6$ intersects itself in figure 2(*b*), so that there is no bifurcation there; the same is true for $M = 5.5$ and, presumably, some other values of M. This semblance of a transcritical bifurcation is merely due to our projecting all flows onto the axis of a single state variable, namely w/R: other state variables of the two solutions at the intersection

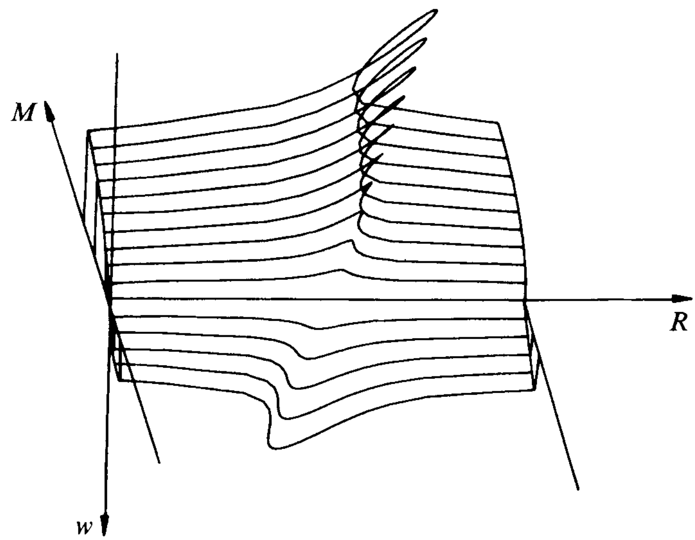

FIGURE 5. A plot of the curves with equation $w = w(R, M)$ for various values of M, as defined by equation (5.9). The branches below the R-axis correspond to $w > 0$ and those above to $w < 0$.

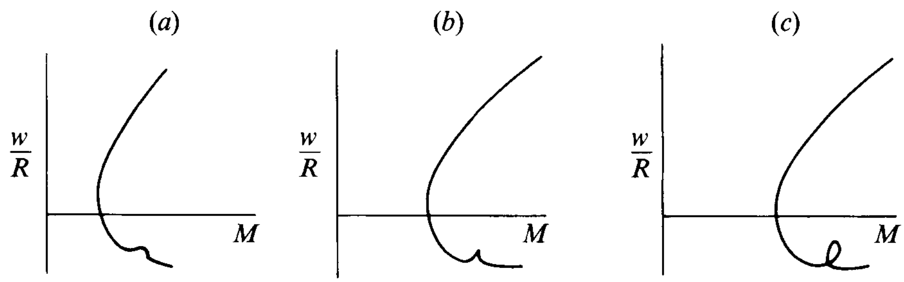

FIGURE 6. Schematic sequence of bifurcation diagrams in the $(M, w/R)$-plane for various values of R: (*a*) $R < R^*$, (*b*) $R = R^*, \approx 17$ (cusp), (*c*) $R > R^*$.

of the curve with itself have different values. So the curves of figure 2(*b*) might be regarded as analogous to different views of a nonplanar smooth wire, the occurrence of the loops and the cusp being merely due to projection onto the planes of view. In figure 4(*a*) we plot, for $M = 6$, w/R versus Rx for $R = 306.72, 39.66, 22.29$. In figure 4(*b*), we plot w/R versus x for $R = 22.29, 19.48, 19.25$. Note the change in the flow pattern in the neighbourhood of the loop in the bifurcation diagram.

Again, it may help to consider a simple model which is *topologically* equivalent to the curves in figure 2(*b*) in order to describe the solution surface in $(R, M, w/R)$-space. The curves *seem* to be equivalent to the the conchoid of Nicomedes, with equation

$$(R - R^*)^2 w^2 = (M^* - M + 1 + w)^2 (1 - w^2). \tag{5.9}$$

For each value of M there are two branches of the curve (5.9), a lower branch for $w > 0$ and an upper branch for $w < 0$ (see figure 5). The curves in figure 2(*a*) which evolve from the Stokes form to the boundary-layer form of type I, as R increases from zero, correspond to the lower branch described by the conchoid equation above, and those in figure 2(*b*) which evolve from the boundary-layer form of type II, as R decreases from infinity, correspond to the upper branch described by the conchoid.

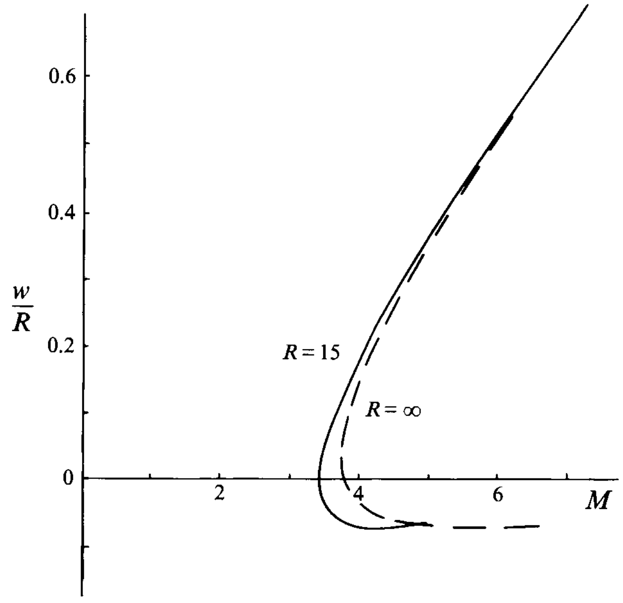

FIGURE 7. The bifurcation diagram in the $(M, w/R)$-plane for $R = 15, \infty$. Note that $w/R = [F'/Rx]_{x=0}, = [f'/2y]_{y=0}$ when $R = \infty$.

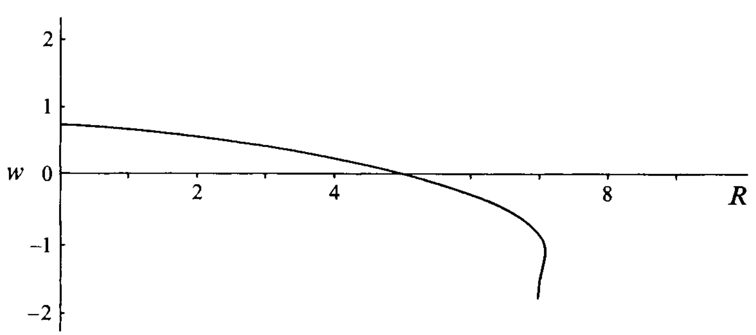

FIGURE 8. Plot of w versus R for $M = 2$.

From the results presented in figure 2 we infer the development of the bifurcation diagrams for constant values of R to be qualitatively that shown in figure 6.

We have recalculated the bifurcation curve of figure 1 for other values of the Reynolds number. The curve for $R = 15$ is shown in figure 7; the curve for $R = \infty$ is repeated for comparison, and it can be seen that the curves are qualitatively similar. For these values of R there is no solution for $M < M_c(R)$ but two solutions, of types I and II, for $M > M_c(R)$; $M_c(15) = 3.43$, whereas $M_c(\infty) = 3.75$. The branch of the curve corresponding to the solution of type II seems for $R = 15$ to have a local maximum at the point $(4.8, -0.06)$; however, the ranges of integration needed for this region of the bifurcation plane are very large, so we have not proceeded with larger values of M.

We have further obtained results for $M < M_c(\infty)$. For $M = 2$ the variation of w/R with R is shown in figure 8; note the turning point at $R \approx 7.1$. We anticipate another turning point at $R \approx 6.95$. The calculations were terminated at $R \approx 6.95$ because again the swirl velocity was becoming very flat at the origin, as if a thin shear layer were about to form away from the origin.

6. Conclusions

This paper is a report of a large number of detailed analytic, asymptotic and numerical results about Long's vortices and their spatial stability. However, we have not yet solved numerically any of the spatial stability problems. So, for the present, we must piece together whatever other evidence is available to infer the stability and instability of the vortices as best we can. First, recall that in general one real temporal eigenvalue increases through zero at a turning point, so that at least one of the two branches which coalesce at a turning point represents unstable flows: we noted this in §1 when introducing Long's results. All the flows are presumably stable in the Stokes limit of small Reynolds number, and both types of flow are unstable (to helical disturbances) as $R \to \infty$. This suggests that flows of type I become stable as R decreases below a critical value which depends on M. The stability of flows of type II is more difficult to discern. For $M < M^*$, when there is no loop of the solution curve in the $(R, w/R)$-plane, the helical instability of flows of type II presumably dies out as R decreases; however, there is no turning point on this branch as R decreases, so it is not clear whether an axisymmetric instability at $R = \infty$ dies out. Again, for $M > M^*$, when the solution curves have loops, a flow of type II has two turning points where there may be exchange of stabilities as R decreases. This presents the possibility of vortex breakdown and hysteresis as R increases and decreases, if at least two of the solutions are stable.

We have noted difficulties in satisfying numerically the boundary conditions at infinity, because the flow there decays algebraically rather than exponentially. This mathematical difficulty is likely to have a physical counterpart. It raises the possibility that conditions quite far from a vortex affect the structure of the vortex significantly, so that in a laboratory experiment the surroundings of the vortex as well as the inlet and outlet conditions may affect the vortex.

Long's vortex and the well-known flows due to a rotating disc provide rare examples of a flow with non-zero helicity density which admits simple mathematical expression, although the helicity density of Long's vortex is singular at the origin $z = 0$ because the velocity field is singular there. We have done no more than evaluate a partial integral of the helicity density in §2.1 and shown that it is not identically zero. It would be more interesting to see how the helicity evolves in time. The conservation of helicity in an inviscid fluid would seem to require that an unstable helical perturbation extracts its helicity from the basic flow because the total helicity of the perturbed flow, basic flow plus perturbation, is conserved.

At a late stage in the preparation of the manuscript of this paper we learned of some independent work of Shtern & Hussain (1993) on a closely related problem. Both they and we have used the same similarity form of solution of the Navier–Stokes equations, but whereas we have followed Long in using cylindrical polar coordinates, Shtern & Hussain have used spherical polar coordinates. The use of the latter leads to a simpler form of the ordinary differential equations, which would appear to be advantageous. However, whereas we, following Long, have imposed the boundary condition $F'(\infty) = 2^{-1/2}$, Shtern & Hussain have imposed (in our notation) $\lim_{x\to\infty} x^{-1}F(x) = 0$. They have also used a slightly different definition of the flow force, M, but both definitions coincide in the boundary-layer limit. Many of our results are qualitatively similar to theirs, but they have not treated stability at all. It follows that the two sets of results may be regarded as complementary, illuminating the rich structures of two closely related families of vortices in a viscous fluid.

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